COMBINATORICA
Akadémiai Kiadó – Springer-Verlag

COMBINATORICA 16 (4) (1996) 505–520

EDGE PARTITIONS OF THE RADO GRAPH

MAURICE POUZET* and NORBERT SAUER**

Received August 29, 1995

We will prove that for every colouring of the edges of the Rado graph, $\mathcal{R}$ (that is the countable homogeneous graph), with finitely many colours, it contains an isomorphic copy whose edges are coloured with at most two of the colours. It was known [4] that there need not be a copy whose edges are coloured with only one of the colours. For the proof we use the lexicographical order on the vertices of the Rado graph, defined by Erdős, Hajnal and Pósa.

Using the result we are able to describe a "Ramsey basis" for the class of Rado graphs whose edges are coloured with at most a finite number, r, of colours. This answers an old question of M. Pouzet.

1. Notation

All graphs $\mathcal{G}$ under consideration will consist of a set $V(\mathcal{G}) = G$ of *vertices* together with a symmetric and reflexive binary relation $\sim$. If $x \sim y$ we will say that the vertex x is *adjacent* to the vertex y. If the vertex x is not adjacent to the vertex y we write $x \not\sim y$. The set of pairs $\{x,y\}$ such that $x \sim y$ and $x \neq y$ is the set of *edges* of $\mathcal{G}$. Note that $\{x,x\}$ is not an edge of $\mathcal{G}$ although $x \sim x$ for all $x \in G$. For $a \in G$ the set of *neighbors* of a is denoted by $\Gamma(a) = \{b \in G\colon\ a \sim b \wedge a \neq b\}$.

The *Rado graph* $\mathcal{R} = (\omega; \sim)$ is the unique countable graph with the property that for every finite graph $\mathcal{G} = (G; \sim)$, vertex $a \in G$ and embedding $\alpha : \mathcal{G} - a \rightarrow \mathcal{R}$ there is an extension of the embedding α to an embedding $\alpha' : \mathcal{G} \rightarrow \mathcal{R}$. (A good source for facts on the Rado graph is [1].) This defining property of the Rado graph is called the *mapping extension property*; it implies that if F is a finite subset of ω and $S \subseteq F$ then there are infinitely many vertices $x \in \omega$ such that x is adjacent to every vertex in S and not adjacent to any vertex in $F - S$.

Let $\mathcal{R} = (\omega; \sim)$ be the Rado graph. A pair $(F; x)$ with $F \cup \{x\} \subseteq \omega$, $x \notin F$ and F finite is a *type* of $\mathcal{R}$, and the *orbit* of the type $(F; x)$ is

$$\mathrm{Orb}(F; x) = \{y \in \omega - F : \forall z \in F\ (z \sim x \Leftrightarrow z \sim y)\}.$$

Mathematics Subject Classification (1991): 05 C 55, 04 A 20

* Supported by the French PRC Math-Info.

**Supported by NSERC of Canada Grant # 691325.

Observe that that $x\in \mathrm{Orb}(F;x)$ and $\mathrm{Orb}(\emptyset;x)=\omega$. It follows from the definition that

$$\forall y \in \mathrm{Orb}(F;x) \quad (\mathrm{Orb}(F;x) = \mathrm{Orb}(F;y)).$$

Note that for $a\in\omega$ we have $\mathrm{Orb}(\{a\};x)=\Gamma(a)$ or $R-(\Gamma(a)\cup\{a\})$ according as $a\sim x$ or $a\not\sim x$. If the vertices y and z are both elements of $\mathrm{Orb}(F;x)$ we say that y and z are of the *same type over* F. The types $(F;x)$ and $(G;y)$ are *disjoint* if $F\cap G=\emptyset$.

Let $\mathcal{G}$ be a countable graph and a a vertex of $\mathcal{G}$ and $\sigma=(\sigma_0,\sigma_1,\sigma_2,\ldots)$ a finite or infinite sequence of vertices of $\mathcal{G}$. We define $\mathrm{lex}(a,\sigma)$, the *lexicographic value* of the vertex a with respect to the sequence σ. If the vertex a is an element of the sequence σ and j is the smallest index such that $a=\sigma_j$ then $\mathrm{lex}(a,\sigma)$ is the sequence $(\varepsilon_i : i\leq j)$ such that

$$\varepsilon_i = \begin{cases} 1 & \text{if } a \sim \sigma_i, \\ 0 & \text{if } a \not\sim \sigma_i. \end{cases}$$

Note that the sequence $\mathrm{lex}(a,\sigma)$ has a "1" in the j-th place because a is adjacent to a. If the vertex a is not an element of the sequence σ let σ' be the sequence obtained from the sequence σ by adding the vertex a as its last element. (We will use this only when σ is a finite sequence; although, the definition has an obvious interpretation for sequences of any order type.) Then $\mathrm{lex}(a,\sigma)=\mathrm{lex}(a,\sigma')$.

For sequences $\sigma=(\sigma_i : i\in\nu)$ and $\tau=(\tau_i : i\in\nu)$, we write $\sigma\preceq\tau$ if and only if σ lexicographically precedes τ. The sequence $(\sigma_i : i\in\nu)$ is an *extension* of the sequence $(\sigma_i : i\in\mu)$ if the ordinal μ is less than or equal to the ordinal ν. The sequence $(\delta_i : i\in\mu)$ is a *subsequence* of $(\sigma_i : i\in\nu)$ if there is an order preserving injection α from μ into ν such that $\forall i\in\mu$ $(\delta_i=\sigma_{\alpha(i)})$. We will only use sequences $(\sigma_i : i\in\nu)$ of distinct vertices of $\mathcal{G}$, $\forall i,j\in\nu$ $(i\neq j\rightarrow\sigma_i\neq\sigma_j)$. The function σ^{-1} is then well defined on the range of σ. It follows easily from the definitions that if a sequence σ' of vertices of $\mathcal{G}$ is an extension of the sequence σ and $\mathrm{lex}(a,\sigma)\prec\mathrm{lex}(b,\sigma)$ then $\mathrm{lex}(a,\sigma')\prec\mathrm{lex}(b,\sigma')$. We will also need the following consequence of the definitions. Let $\sigma=(\sigma_i : i<n)$ be a finite sequence of vertices of $\mathcal{G}$ and let $\sigma'=(\sigma_i : i\leq n)$ be an extension of σ with σ_n a vertex of $\mathcal{G}$. If $a\in G$ is not an element of σ' and $\mathrm{lex}(a,\sigma)=\mathrm{lex}(\sigma_n,\sigma)$, then $a\not\sim\sigma_n$ implies that $\mathrm{lex}(a,\sigma')\prec\mathrm{lex}(\sigma_n,\sigma')$ and $a\sim\sigma_n$ implies that $\mathrm{lex}(a,\sigma')\succ\mathrm{lex}(\sigma_n,\sigma')$.

Let $\mathcal{R}=(\omega;\sim)$ be the Rado graph and let σ be any permutation of ω. Thus $(\sigma_0,\sigma_1,\ldots)$ is an enumeration of the vertices of $\mathcal{R}$. Define an order relation $\leq_\sigma$ on ω by

$$a \leq_\sigma b \Leftrightarrow \mathrm{lex}(a,\sigma) \preceq \mathrm{lex}(b,\sigma),$$

We call $\leq_\sigma$ a *lexicographic order*, and denote the structure $(\omega;\sigma,\sim,<_\sigma)$ by $\mathcal{R}_\sigma$. $\mathcal{R}_\sigma$ is called a *lexicographically ordered Rado graph.* The order $<_\sigma$ defines a partition of the set of edges of $\mathcal{R}$, $E=E(<_\sigma)\cup E(>_\sigma)$, where

$$E(<_\sigma) = \{\{\sigma_i,\sigma_j\} : i<j \wedge \sigma_i \sim \sigma_j \wedge \sigma_i <_\sigma \sigma_j\},$$

$$E(>_\sigma) = \{\{\sigma_i,\sigma_j\} : i<j \wedge \sigma_i \sim \sigma_j \wedge \sigma_i >_\sigma \sigma_j\}.$$

Let σ and τ be two permutations of ω. An *embedding* of $\mathcal{R}_\tau$ into $\mathcal{R}_\sigma$ is a map $\alpha^*:\omega\to\omega$ induced by a strictly increasing map $\alpha:\omega\to\omega$ via $\alpha^*(\tau_i)=\sigma_{\alpha(i)}$ such that

$$\forall i,j\in\omega(\tau_i\sim\tau_j\Leftrightarrow\sigma_{\alpha(i)}\sim\sigma_{\alpha(j)}\wedge\tau_i\leq_\tau\tau_j\Leftrightarrow\sigma_{\alpha(i)}\leq_\sigma\sigma_{\alpha(j)}).$$

Thus α^* is an injective map from $\mathcal{R}_\tau$ into $\mathcal{R}_\sigma$ which preserves the graph structure, the order of enumeration and the lexicographic order. We denote by $\alpha^*(\mathcal{R}_\tau)$, the copy of $\mathcal{R}_\tau$ in $\mathcal{R}_\sigma$ *i.e.* the restriction $\mathcal{R}_\sigma|\{\sigma_{\alpha(i)}:i\in\omega\}$. Note that $\alpha^*(E(<_\tau)\subseteq E(<_\sigma)$ and $\alpha^*(E>_\tau)\subseteq E(>_\sigma)$. Although $\mathcal{R}_\sigma$ depends upon σ, it will turn out, as a consequence of Lemma 7, that any two lexicographically ordered Rado graphs are equimorphic; that is, there are embeddings from each one into the other. A *type* of $\mathcal{R}_\sigma$ is a type of the underlying Rado graph $\mathcal{R}=(\omega;\sim)$.

We shall not use the fact, but it is interesting to observe the following fact:

$\leq_\sigma$ is a dense linear order on ω without a maximum or minimum.

Proof. Let $a=\sigma_i$ and $b=\sigma_j$ be two vertices of $\mathcal{R}$ such that $a<_\sigma b$. Because of the mapping extension property there is a vertex $c=\sigma_m\in\omega$ such that $m>\max\{i,j\}$ and for $k\leq\max\{i,j\}$

$$c\sim\sigma_k\Leftrightarrow k\leq i\wedge a\sim\sigma_k.$$

Clearly $\operatorname{lex}(a,\sigma)\prec_\sigma\operatorname{lex}(c,\sigma)\prec_\sigma\operatorname{lex}(b,\sigma)$ and hence $a<_\sigma c<_\sigma b$. Similarly it can be seen that there is no $\leq_\sigma$ maximum or minimum. ∎

Let T be a set and $\leq$ an order relation on T. For sets $A,B\subseteq T$ we write $A<B$ if, for all $a\in A$ and all $b\in B$, $a<b$ holds. Of course $a<b$ if $a\leq b$ and $a\neq b$.

Assume that σ and τ are two permutations of ω. We will write

$$\mathcal{R}_\sigma\to(\mathcal{R}_\tau)_r^<\tag{1}$$

to mean that for every colouring f of the edges of $E(<_\sigma)$ with r colours there is an embedding α^* of $\mathcal{R}_\tau$ into $\mathcal{R}_\sigma$ such that f is constant on $\alpha^*(E(<_\tau))$. The relation

$$\mathcal{R}_\sigma\to(\mathcal{R}_\tau)_r^>\tag{2}$$

is defined in a similar way for colourings of $E(>_\sigma)$.

If the edges of the lexicographically ordered Rado graph $\mathcal{R}_\sigma$ are coloured with two colours c_0 and c_1 (not necessarily different) such that every edge in $E(<_\sigma)$ is coloured with colour c_0 and every edge in $E(>_\sigma)$ has colour c_1 then this edge-coloured Rado graph is called a *basic coloured Rado graph* or, more precisely, a *basic* (c_0,c_1)*-coloured Rado graph*. The colouring of a basic coloured Rado graph is a *basic colouring*. From a set of r colours there are r^2 basic colourings of the edges of $\mathcal{R}_\sigma$. We wish also to be able to speak of a basic coloured Rado graph $\mathcal{R}=(\omega;\sim)$ in the case where the vertices ω of $\mathcal{R}$ are not ordered. All we mean by this is that there exists a permutation $\sigma:\omega\to\omega$ such that the colouring of the corresponding lexicographically ordered Rado graph $\mathcal{R}_\sigma$ is a basic colouring.

2. Introduction

In [2] S. Todorcevic describes several results of the following nature. "Given is some structure A and a mapping f defined on some cube A^r or $[A]^r$ of A and we need to find a large $B \subseteq A$ where the behavior of f is as simple as possible". For example there is a function f from $[\mathbf{Q}]^2$ to 2 such that for every order isomorphic copy $\mathbf{Q}_0$ of $\mathbf{Q}$ in $\mathbf{Q}$, $|f([\mathbf{Q}_0]^2)| = 2$. On the other hand, for every $r \in \omega$ and function f from $[\mathbf{Q}]^2$ to r, there is a copy $\mathbf{Q}_0$ of $\mathbf{Q}$ in $\mathbf{Q}$ such that $|f([\mathbf{Q}_0]^2)| \leq 2$. This is a famous unpublished result of F. Galvin, see [3]. In the present paper we present a similar result for the Rado graph. Our main result, Theorem 5.2, says that if $\mathcal{R}_\sigma$ and $\mathcal{R}_\tau$ are two lexicographically ordered Rado graphs then the relations (1) and (2) hold. This implies that if the edges of $\mathcal{R}$ are coloured with r colours, then $\mathcal{R}$ contains an isomorphic copy in which the edges have at most two colours.

Let $\mathcal{R} = (\omega; \sim)$ be the Rado graph with ε the identity permutation on ω and $\mathcal{R}_\varepsilon = (\omega; \varepsilon, \sim, <_\varepsilon)$. It has been noticed by Erdős, Hajnal and Pósa in [4] that $\mathcal{R} \not\to (\mathcal{R})^2_2$. For completeness sake we will prove this assertion which follows from the next lemma because every countable graph can be embedded into the Rado graph, (see [1]).

Lemma 2.1. *The Rado graph $\mathcal{R}$ does not contain a complete bipartite graph $K_{\omega,\omega}$ with all its edges in $E_{<_\varepsilon}$ or all its edges in $E_{>_\varepsilon}$.*

Proof. Let $A = (a_i; i \in \omega)$ and $B = (b_i; i \in \omega)$ be two disjoint sets of vertices of $\mathcal{R}$ such that for all $i, j \in \omega$ the pair (a_i, b_j) is an edge but the pairs (a_i, a_j) and (b_i, b_j) are not edges of $\mathcal{R}$.

Case 1: All the edges of the bipartite graph (A, B) are elements of $E_{<_\varepsilon}$. Let σ be the sequence $0, 1, 2, \ldots, a_0 - 1$. Let $n \in \omega$ be such that for all $i \in \omega$ with $b_i > a_0$ the sequence $\mathrm{lex}(b_n, \sigma)$ is maximal in the lexicographic ordering. There is $m \in \omega$ such that $b_n < a_m$. The edge $(b_n, a_m) \in E_{<_\varepsilon}$ and b_n is, but a_m is not adjacent to a_0. This implies that $\mathrm{lex}(a_m, \sigma)$ is strictly larger than $\mathrm{lex}(b_n, \sigma)$ in the lexicographic ordering. There is $l \in \omega$ such that $a_m < b_l$. This implies that $\mathrm{lex}(b_l, \sigma)$ is larger than or equal to $\mathrm{lex}(a_m, \sigma)$ in the lexicographic order and hence strictly larger than $\mathrm{lex}(b_n, \sigma)$ contrary to the choice of n.

Case 2: All the edges of the bipartite graph (A, B) are elements of $E_{>_\varepsilon}$. Let σ be the sequence $0, 1, 2, \ldots, a_0 - 1$. Let $n \in \omega$ be such that for all $i \in \omega$ with $a_i > a_0$ the sequence $\mathrm{lex}(b_n, \sigma)$ is minimal in the lexicographic ordering. There is $m \in \omega$ such that $a_n < b_m$. The edge $(a_n, b_m) \in E_{>_\varepsilon}$ and a_n is not, but b_m is adjacent to a_0. This implies that $\mathrm{lex}(b_m, \sigma)$ is strictly smaller than $\mathrm{lex}(a_n, \sigma)$ in the lexicographic ordering. There is $l \in \omega$ such that $b_m < a_l$. This implies that $\mathrm{lex}(a_l, \sigma)$ is smaller than or equal to $\mathrm{lex}(b_m, \sigma)$ and hence strictly smaller than $\mathrm{lex}(a_n, \sigma)$ contrary to the choice of n. ∎

Lemma 2.2. *The vertices of every finite graph $\mathscr{G}$ have an ordering $\sigma = (a_0, a_1, \dots, a_{n-1})$ such that whenever $0 \le i < j < n$ then $\operatorname{lex}(a_i, \sigma)$ is smaller than $\operatorname{lex}(a_j, \sigma)$ in the lexicographic order. Also, the vertices of every finite graph $\mathscr{G}$ have an ordering $\sigma = (a_0, a_1, \dots, a_{n-1})$ such that whenever $0 \le i < j < n$ then $\operatorname{lex}(a_i, \sigma)$ is larger than $\operatorname{lex}(a_j, \sigma)$ in the lexicographic order.*

Proof. By induction on n. Choose any vertex of $\mathscr{G}$ for a_0. Then order the vertices not adjacent to a_0 to obtain the sequence $a_0, a_1, \dots, a_i$ for some $i < n$. Then order the vertices adjacent to a_0 and concatinate with the sequence $a_0, a_1, \dots, a_i$.

To prove the second statement choose any vertex of $\mathscr{G}$ for a_0. Then order the vertices adjacent to a_0 to obtain the sequence $a_0, a_1, \dots, a_i$ for some $i < n$. Then order the vertices not adjacent to a_0 and concatinate with the sequence $a_0, a_1, \dots, a_i$. ∎

Lemma 2.3. *If $\mathscr{G}$ is a finite graph then there is an embedding α from $\mathscr{G}$ into $\mathscr{R}$ such that all of the edges of $\alpha(\mathscr{G})$ are in $E_{<_\varepsilon}$ and an embedding β from $\mathscr{G}$ into $\mathscr{R}$ such that all of the edges of $\beta(\mathscr{G})$ are in $E_{>_\varepsilon}$.*

Proof. Order the vertices of $\mathscr{G}$ into a sequence $\sigma = (a_0, a_1, \dots, a_{n-1})$ such that whenever $0 \le i < j < n$ then $\operatorname{lex}(a_i, \sigma)$ is smaller than $\operatorname{lex}(a_j, \sigma)$ in the lexicographic order. Choose an arbitrary image $\alpha(a_0)$.

Assume $\alpha(a_0) < \alpha(a_1) < \cdots < \alpha(a_i)$ has already been determined. According to the mapping extension property of $\mathscr{R}$ there is an element $m \in \omega$ such that $\alpha(a_i) < m$ and if $l \le \alpha(a_i)$ with $l \sim m$ then there is $j < n$ such that $a_j \sim a_{i+1}$. Put $\alpha(a_{i+1}) = l$. It is easy to see that the embedding α so constructed has the required properties.

To construct the embedding β change the construction above apropriately. ∎

Using the fact that every countable graph can be embedded into the Rado graph the following theorem follows easily from [4, Theorem 1]: *If the edges of the Rado graph $\mathscr{R}$ are coloured with two colours red and blue and there is a finite graph $\mathscr{G}$ such that every copy of $\mathscr{G}$ in $\mathscr{R}$ contains a blue edge then there is a copy of $\mathscr{R}$ in $\mathscr{R}$ which has only blue edges.*

This theorem is a consequence of (1) and (2) and hence of Theorem 5.2 as follows. Colour the edges of $\mathscr{R}$ red and blue and assume that there is a finite graph $\mathscr{G}$ such that every copy of $\mathscr{G}$ in $\mathscr{R}$ contains a blue edge. Using (1) there is an embedding α from $\mathscr{R}_\varepsilon$ into $\mathscr{R}_\varepsilon$ such that $\alpha(E_{<_\varepsilon})$ is monochromatic. Applying (2) to this copy $\alpha(\mathscr{R})$ we obtain an embedding β from $\mathscr{R}_\varepsilon$ into $\mathscr{R}_\varepsilon$ such that $\beta(E_{<_\varepsilon})$ and $\beta(E_{>_\varepsilon})$ are monochromatic. The graph $\mathscr{G}$ can be so embedded into $\beta(\mathscr{R})$ such that all of the edges of this embedding are edges of $\beta(E_{<_\varepsilon})$ and so that all of the edges of the embedding are edges of $\beta(E_{>_\varepsilon})$. This implies that both sets of edges $\beta(E_{<_\varepsilon})$ and $\beta(E_{>_\varepsilon})$ are monochromatically blue and hence that all of the edges of $\beta(\mathscr{R})$ are blue.

Another way of looking at our result is provided by Theorem 5.3. Assume that $f : [\omega]^2 \to r \in \omega$ is a colouring of all the two element subsets of vertices of $\mathscr{R}$. Then $\mathscr{R}$ contains an induced copy $\mathscr{R}_0$ of itself such that $|f([R_0]^2| \le 4$. In fact, we can precisely specify a partition of $[\mathscr{R}_0]^2$ into four sets, on each of which f is constant. These are

$$S_1 = \{(\sigma_i, \sigma_j) : \sigma_i, \sigma_j \in \mathscr{R}_0 \wedge \sigma_i \sim \sigma_j \wedge \sigma_i <_\varepsilon \sigma_j \wedge \sigma_i <_\sigma \sigma_j \wedge \sigma_i < \sigma_j\},$$
$$S_2 = \{(\sigma_i, \sigma_j) : \sigma_i, \sigma_j \in \mathscr{R}_0 \wedge \sigma_i \sim \sigma_j \wedge \sigma_i <_\varepsilon \sigma_j \wedge \sigma_i <_\sigma \sigma_j \wedge \sigma_i > \sigma_j\},$$
$$S_3 = \{(\sigma_i, \sigma_j) : \sigma_i, \sigma_j \in \mathscr{R}_0 \wedge \sigma_i \not\sim \sigma_j \wedge \sigma_i <_\varepsilon \sigma_j \wedge \sigma_i <_\sigma \sigma_j \wedge \sigma_i < \sigma_j\},$$
$$S_4 = \{(\sigma_i, \sigma_j) : \sigma_i, \sigma_j \in \mathscr{R}_0 \wedge \sigma_i \not\sim \sigma_j \wedge \sigma_i <_\varepsilon \sigma_j \wedge \sigma_i <_\sigma \sigma_j \wedge \sigma_i > \sigma_j\},$$

where σ is an ω ordering of $\mathscr{R}_0$.

Let $\mathbf{R}_r$ be the set of all Rado graphs defined on ω with edges coloured with at most r colours. For $\mathscr{R}$, $\mathscr{S} \in \mathbf{R}_r$ an injection $\alpha : \mathscr{R} \to \mathscr{S}$ is an *embedding* if it respects the adjacency relation and the colour. If there is an embedding from $\mathscr{R}$ to $\mathscr{S}$ we will write $\mathscr{R} \to \mathscr{S}$, and $\mathscr{R} \not\to \mathscr{S}$ denotes the negation. The coloured Rado graphs $\mathscr{R}$ and $\mathscr{S}$ are *equimorphic* if $\mathscr{R} \to \mathscr{S} \wedge \mathscr{S} \to \mathscr{R}$. Note that the equivalence classes of equimorphic coloured Rado graphs of $\mathbf{R}_r$ form a partial order under $\to$. The subset $\mathbf{B}_r \subseteq \mathbf{R}_r$ is a *Ramsey basis* of $\mathbf{R}_r$ if

$$\forall \mathscr{R} \in \mathbf{R}_r \; \exists \mathscr{B} \in \mathbf{B}_r \quad (\mathscr{B} \to \mathscr{R}) \quad \text{and} \quad \forall \mathscr{R}, \mathscr{S} \in \mathbf{B}_r \; (\mathscr{R} \neq \mathscr{S} \to \mathscr{R} \not\to \mathscr{S})).\}$$

Obviously, this definition of a Ramsey basis of the set of coloured Rado graphs can be generalized to other relational structures. For example, in the set of all countable complete graphs whose edges are coloured with $r \in \omega$ colours, the r monochromatically edge coloured complete graphs form a Ramsey basis.

M. Pouzet asked to find a Ramsey basis for $\mathbf{R}_r$ ordered by $\to$, and Theorem 5.4 answers this question. Note that knowledge of such a basis provides complete information for an understanding of the edge partition problem of the Rado graph. For $r \in \omega$ and every pair $c_0, c_1 \in r$ with $c_0 \leq c_1$, choose a (c_0, c_1)-coloured basic Rado graph $\mathscr{B}_{(c_0,c_1)}$. Theorem 5.4 says that the set $\mathbf{B}_r = \{\mathscr{B}_{(c_0,c_1)} : c_0, c_1 \in r \wedge c_0 \leq c_1\}$ is a Ramsey basis of the set $\mathbf{R}_r$. In the case of Rado graphs with edges coloured red and blue, we know from the example in [4] that there are more minimal elements under embedding than the monochromatic ones. It follows from Theorem 5.4, that the two monochromatically coloured Rado graphs together with a (red, blue)-coloured basic Rado graph form a Ramsey basis. Hence every edge coloured Rado graph, using colours red and blue, either contains a copy of the Rado graph in which all edges are coloured red, or a copy in which all edges are coloured blue, or a (red, blue)-coloured basic Rado graph.

3. Scattered sets

In this section σ denotes a fixed permutation of ω and $\mathscr{R}_\sigma$ is the corresponding lexicographically ordered Rado graph. As usual, for $a \in \omega$ and $F \subseteq \omega$ we denote by $\sigma^{-1}(a)$ the index i such that $a = \sigma_i$ and $\sigma^{-1}(F) = \{\sigma^{-1}(a) : a \in F\}$. Put

$\mathscr{E} = \{\mathrm{lex}(a,\sigma) : a \in \omega\}$. Note that the elements of $\mathscr{E}$ are finite 0,1-sequences which end in the number 1 and that the function $\mathrm{lex}(*,\sigma)$ is a bijection from ω to $\mathscr{E}$. For two elements ε_0 and ε_1 of $\mathscr{E}$ we write $\varepsilon_0 \sqsubseteq \varepsilon_1$ if ε_1 is an extension of the sequence ε_0, and for $\varepsilon \in \mathscr{E}$ we define $\mathrm{ext}(\varepsilon) = \{\delta \in \mathscr{E} : \varepsilon \sqsubseteq \delta\}$. Pulling this notation back to ω we define $a \sqsubseteq b$ for a, $b \in \omega$, to mean that $\mathrm{lex}(a,\sigma) \sqsubseteq \mathrm{lex}(b,\sigma)$, and we define $\mathrm{ext}(a) = \{b \in \omega : a \sqsubseteq b\}$. It follows from the definition of $\sqsubseteq$ that $a \sqsubseteq b$ if and only if the vertices a and b are of the same type over the set $\{\sigma_i : i < \sigma^{-1}(a)\}$ and $a \sim b$.

We define an ideal $\mathscr{I}$ of subsets of ω. The set $S \subseteq \omega$ is *small* if

$$\forall a \in \omega\ \exists b \sqsupseteq a\ (\mathrm{ext}(b) \cap S = \emptyset),$$

and $\mathscr{I}$ is the set of small subsets of ω. A subset of ω which is not small is *large*, and we denote the set of large subsets of ω by $\mathscr{L}$. Note that $L \in \mathscr{L}$ if and only if

$$\exists a \in \omega\ \forall b \sqsupseteq a\ (\mathrm{ext}(b) \cap L \neq \emptyset).$$

Remark. Note that $\mathrm{ext}(a) \in \mathscr{L}$ for any $a \in \omega$. Also, if a set $L \subseteq \omega$ is dense in the order $<_\sigma$, then it is large. Is it true that a set L is large if and only if L contains a $<_\sigma$-dense subset, in other words are the small sets just the $<_\sigma$-scattered sets?

Lemma 3.1. *Every finite subset of ω is small.*

Proof. Note that if $x \sqsupseteq y$ then $\sigma^{-1}(x) \geq \sigma^{-1}(y)$. Let $F \subseteq \omega$ be finite and let $a \in \omega$. There are infinitely many $x \in \omega$ such that $x \sqsupseteq a$. Hence there is a $b \sqsupseteq a$ with $\sigma^{-1}(b) > \max \sigma^{-1}(F)$. Then $\mathrm{ext}(b) \cap F = \emptyset$. ∎

Lemma 3.2. *$\mathscr{I}$ is a proper ideal.*

Proof. Let $T \subseteq S \in \mathscr{I}$ and let $a \in \omega$. Then there exists $b \sqsupseteq a$ such that $\mathrm{ext}(b) \cap S = \emptyset$ which implies that $\mathrm{ext}(b) \cap T = \emptyset$, and hence $T \in \mathscr{I}$.

Let $S, T \in \mathscr{I}$, and let $a \in \omega$. Then there are b, $c \in \omega$ such that $c \sqsupseteq b \sqsupseteq a$ and $\mathrm{ext}(b) \cap S = \emptyset$, $\mathrm{ext}(c) \cap T = \emptyset$. Then $\mathrm{ext}(c) \cap (S \cup T) = \emptyset$, and so $S \cup T \in \mathscr{I}$.

The set ω is not small since, for every $b \in \omega$, the set $\mathrm{ext}(b) \cap \omega \neq \emptyset$ since $b \in \mathrm{ext}(b)$. ∎

Lemma 3.3. *If $(F;x)$ is a type of $\mathscr{R}$ and $a \in \omega$ with $\min(\sigma^{-1}(F)) > \sigma^{-1}(a)$ then there is $a_0 \sqsupseteq a$ such that $\mathrm{ext}(a_0) \subseteq \mathrm{Orb}(F;x)$.*

Proof. Choose an element $a_0 \in \mathrm{Orb}(F;x)$ with $\sigma^{-1}(a_0) > \max(\sigma^{-1}(F))$ and such that $\forall i \leq \max(\sigma^{-1}(F))$:

$$a_0 \sim \sigma_i \quad \text{iff} \quad \begin{cases} a \sim \sigma_i, & \text{for } 0 \leq i \leq \sigma^{-1}(a), \\ \sigma_i \notin F & \text{for } \sigma^{-1}(a) < i \leq \max(\sigma^{-1}(F)), \\ x \sim \sigma_i, & \text{for } \sigma_i \in F. \end{cases}$$

Due to the mapping extension property of the Rado graph there are infinitely many such elements a_0. Clearly $a_0 \sqsupseteq a$ and $b \in \mathrm{Orb}(F;x)$ if $b \sqsupseteq a_0$. ∎

Lemma 3.4. *If $S \subseteq \omega$ and $((F_i;x_i):i\in\omega)$ is an ω-sequence of pairwise disjoint types of $\mathcal{R}$ such that*

$$\forall i \in \omega \ (S \cap \operatorname{Orb}(F_i;x_i) \text{ is small})$$

then $S \in \mathcal{I}$.

Proof. We can assume that the $F_i \neq \emptyset$, otherwise $S = S \cap \operatorname{Orb}(F_i,x_i) \in \mathcal{I}$. Let $a \in \omega$. The types of the sequence $((F_i;x_i):i\in\omega)$ are pairwise disjoint and hence there is $j \in \omega$ such that $\min \sigma^{-1}(F_j) > \sigma^{-1}(a)$. Using Lemma 3.3 there is an $a_0 \sqsupseteq a$ such that $\operatorname{ext}(a_0) \subseteq \operatorname{Orb}(F_j;x_j)$. The set $S \cap \operatorname{Orb}(F_j;x_j)$ is small and hence there is a $b \sqsupseteq a_0$ such that $\operatorname{ext}(b) \cap (S \cap \operatorname{Orb}(F_j;x_j)) = \emptyset$. Because $b \sqsupseteq a_0$ and $\operatorname{ext}(a_0) \subseteq \operatorname{Orb}(F_j;x_j)$ also $\operatorname{ext}(b) \subseteq \operatorname{Orb}(F_j;x_j)$, and hence $\operatorname{ext}(b) \cap (S \cap \operatorname{Orb}(F_j;x_j)) = \operatorname{ext}(b) \cap S = \emptyset$ and so S is small. ∎

Corollary 3.5. *If $A \subseteq \omega$ and there are infinitely many vertices $b \in \omega$ for which the set $\Gamma(b) \cap A$ is small, or if there are infinitely many vertices $b \in \omega$ for which the set $A \setminus \Gamma(b)$ is small, then the set A is small.* ∎

Lemma 3.6. *For every large set $L \in \mathcal{L}$ there are two large sets $L_0, L_1 \in \mathcal{L}$ such that $L_0 <_\sigma L_1$ and $L_0 \cup L_1 \subseteq L$.*

Proof. Since $L \in \mathcal{L}$ there is an $a \in \omega$ such that for all $b \sqsupseteq a$ the set $\operatorname{ext}(b) \cap L$ is not empty. Let the vertices $a_0, a_1, x \in \omega$ be such that $\sigma^{-1}(a_1) > \sigma^{-1}(a_0) > \sigma^{-1}(x) = \sigma^{-1}(a) + 1$ and $a_1 \sqsupseteq a$ and $a_0 \sqsupseteq a$ and $a_1 \sim x$ and $a_0 \not\sim x$. Such vertices a_1 and a_0 exist because of the mapping extension property of the Rado graph. Then $L_0 = \operatorname{ext}(a_0) <_\sigma L_1 = \operatorname{ext}(a_1)$ and L_0, L_1 are both large. ∎

Lemma 3.7. *Let $\mathcal{B}$ be a finite set of large sets. Then there exists to each $B \in \mathcal{B}$ a large set $B' \subseteq B$ such that, for all $A, B \in \mathcal{B}$ and all $x \in B'$, the sets $A' \cap \Gamma(x)$ and $A' \setminus \Gamma(x)$ are large.*

Proof. For A, $B \in \mathcal{B}$ let

$$S_{AB} = \{x \in B : \Gamma(x) \cap A \in \mathcal{I}\},$$
$$T_{AB} = \{x \in B : A - \Gamma(x) \in \mathcal{I}\}.$$

It follows from Corollary 3.5 that the set

$$S = \bigcup_{A,B \in \mathcal{B}} (S_{AB} \cup T_{AB})$$

is finite. Then, for each $B \in \mathcal{B}$, the set $B' = B \setminus S$ is large, and for $x \in B'$ the sets $A' \cap \Gamma(x)$ and $A' \setminus \Gamma(x)$ are large. ∎

4. The embedding construction

Let $\sigma=(\sigma_i : i\in\omega)$ be a permutation of ω and let $\mathcal{R}_\sigma=(\omega;\sigma,\sim,\leq_\sigma)$ be the corresponding lexicographically ordered Rado graph. Suppose that $\delta=(\delta_i; i\in\omega)$ is a subsequence of the sequence σ. We will then denote by $\leq_\delta$ the lexicographic order of ω generated by δ. Thus, if $a,b\in\omega$, then $a\leq_\delta b$ holds if $\mathrm{lex}(a,\delta)\preceq\mathrm{lex}(b,\delta)$. The subsequence δ of σ is an *order preserving subsequence* of σ if, for all $i,\ j\in\omega$ $(\delta_i\leq_\delta\delta_j\Leftrightarrow\delta_i\leq_\sigma\delta_j)$.

Let τ be another permutation of ω. Suppose that δ is a subsequence of σ and that the function $\alpha:\omega\to\omega$ given by $\forall i\in\omega\ (\alpha(\tau_i)=\delta_i)$ is a graph isomorphism, that is $\forall i,\ j\in\omega\ (\tau_i\sim\tau_j\Leftrightarrow\delta_i\sim\delta_j)$. It follows that the subsequence δ is order preserving if and only if the function α is order preserving, because $\forall i,\ j\in\omega\ (\tau_i\leq_\tau\tau_j\Leftrightarrow\delta_i\leq_\sigma\delta_j)$.

Lemma 4.1. *Let $\mathcal{R}_\sigma$ and $\mathcal{R}_\tau$ be lexicographically ordered Rado graphs. Then, for every colouring f of the edges of $E(<_\sigma)$ with colours 0 and 1, there exists a colour $\ell\in 2$ and an order preserving subsequence δ of σ such that:*

(i) *the function α given by $\alpha(\tau_i)=\delta_i$ is an order preserving graph isomorphism;*

(ii) $\forall i,\ j\in\omega\ (\delta_i\sim\delta_j\wedge\delta_i<_\sigma\delta_j\wedge i<j\to f(\{\delta_i,\delta_j\})=\ell)$.

Proof. For $\ell\in 2$ and $a\in\omega$ define $\Gamma_\ell(a)=\{b\in R: b\sim a\wedge a\neq b\wedge f(\{b,a\})=\ell\}$, and for $A,B\subseteq\omega$ let $\phi_l(A,B)$ denote the assertion

$$\forall A'\subseteq A\ \forall B'\subseteq B\quad (A',\ B'\in\mathcal{L}\Rightarrow\{a\in A':\Gamma_\ell(a)\cap B'\in\mathcal{L}\}\in\mathcal{L}).$$

Note that $\phi_l(A,B)\Rightarrow\phi_l(A',B')$ whenever $A'\subseteq A$ and $B'\subseteq B$ are large. We write ψ_ℓ for the formula:

$$\exists U\in\mathcal{L}\ \forall V\subseteq U\ (V\in\mathcal{L}\Rightarrow\ \exists A,\ B\subseteq V\ (A,B\in\mathcal{L}\wedge A<_\sigma B\wedge\phi_\ell(A,B))).$$

Claim 1. *If A,B are large then there is $\ell\in 2$ such that $\{a\in A:\Gamma_\ell(a)\cap B\in\mathcal{L}\}\in\mathcal{L}$.*

Proof of Claim. Suppose that $K=\{a\in A:\Gamma_0(a)\cap B\in\mathcal{L}\}\in\mathcal{J}$. Since B is large, it follows by Corollary 3.5 that $S=\{a\in A:\Gamma(a)\cap B\in\mathcal{J}\}\in\mathcal{J}$. Hence $A_1=A\setminus(K\cup S)\in\mathcal{L}$. For $a\in A_1$ we have that $\Gamma(a)\cap B\in\mathcal{L}$ and $\Gamma_0(a)\cap B\in\mathcal{J}$ and hence $\Gamma_1(a)\cap B\in\mathcal{L}$. Therefore, $A_1\subseteq\{a\in A:\Gamma_1(a)\cap B\in\mathcal{L}\}\in\mathcal{L}$. ∎

Claim 2. *Either ψ_0 or ψ_1 holds.*

Proof of Claim. $\neg\psi_0$ implies that for any large set M there is a large subset $N\subseteq M$ such that

$$\forall X,\ Y\subseteq N\ (X,Y\in\mathcal{L}\wedge X<_\sigma Y\Rightarrow\neg\phi_0(X,Y)).$$

In particular, for $M=\omega$ there is a large set U such that the above holds with $N=U$. Consider any large subset $V\subseteq U$. By Lemma 3.6 there are large sets $A,\ B\subseteq V$ such that $A<_\sigma B$. Let $A'\subseteq A$ and $B'\subseteq B$ be large. Then $A'<_\sigma B'$. Letting A',B' play the role of X, Y in the sentence above, it follows that $\neg\phi_0(A', B')$ holds. Hence there are large sets $A'' \subseteq A'$ and $B'' \subseteq B'$ such that $\{a \in A'' : \Gamma_0(a) \cap B'' \in \mathscr{L}\} \in \mathscr{I}$. Then $\{a \in A'' : \Gamma_1(a) \cap B'' \in \mathscr{L}\} \in \mathscr{L}$ by Claim 1. This implies that the set $\{a \in A' : \Gamma_1(a) \cap B' \in \mathscr{L}\}$ is also large. In other words, $\phi_1(A, B)$ holds. Since V was an arbitrary large subset of U, this shows that ψ_1 holds. ∎

In view of the last claim, we may assume that ψ_0 holds. We shall construct a subsequence $\delta = (\delta_0, \delta_1, \delta_2, \ldots)$ of $\sigma = (\sigma_0, \sigma_1, \sigma_2, \ldots)$ such that

(o) the function α given by $\alpha(\tau_i) = \delta_i$ is an order preserving graph isomorphism, that is $\tau_i \sim \tau_j \to \delta_i \sim \delta_j$ and $\tau_i \leq_\tau \tau_j \to \delta_i \leq_\sigma \delta_j$,

(oo) $\forall i, j \in \omega\ (\delta_i \sim \delta_j \wedge \delta_i <_\sigma \delta_j \wedge i < j \to f(\{\delta_i, \delta_j\})) = 0$.

Let U be a large set which witnesses ψ_0. We shall inductively define elements $\delta_n \in U$ and a finite set $\mathscr{B}_n$ of large subsets of U. Let $\mathscr{B}_0 = \{U\}$. Suppose $n \in \omega$ and that we have already defined $\mathscr{B}_n$ and $Y_n = \{\delta_i : i < n\}$ so that the conditions (i)–(viii) below are satisfied. Let δ^n be the sequence $(\delta_0, \delta_1, \delta_2, \ldots, \delta_{n-1})$ and let $\leq_n$ be the lexicographic order on ω induced by the sequence δ^n; that is $a \leq_n b$ if $\operatorname{lex}(a, \delta^n) \preceq \operatorname{lex}(b, \delta^n)$.

(i) The sequence δ^n is a subsequence of the sequence σ.

(ii) $\mathscr{B}_n$ is a finite set of pairwise disjoint large subsets of U and $\min(\sigma^{-1}(B)) > \max(\sigma^{-1}(Y_n))$ for each set $B \in \mathscr{B}_n$. Also, for every type of the form $(Y_n; x)$, there is exactly one set $B \in \mathscr{B}_n$ such that $B \subseteq \operatorname{Orb}(Y_n; x)$.

(iii) The set $\mathscr{B}_n \cup Y_n$ is totally ordered under $\leq_\sigma$ and $\leq_n$, and these two orders agree on $\mathscr{B}_n \cup Y_n$.

(iv) The function $\alpha_n : \{\tau_0, \tau_1, \ldots, \tau_{n-1}\} \to \{\delta_0, \delta_1, \ldots, \delta_{n-1}\}$ given by $\alpha(\tau_i) = \delta_i$ is an order preserving graph isomorphism, that is $\tau_i \sim \tau_j \Rightarrow \delta_i \sim \delta_j$ and $\tau_i \leq_\tau \tau_j \Rightarrow \delta_i \leq_\sigma \delta_j$.

(v) $\forall i, j \in n\ (\delta_i \sim \delta_j \wedge \delta_i <_\sigma \delta_j \wedge i < j \Rightarrow f(\{\delta_i, \delta_j\}) = 0)$.

(vi) $\forall A, B \in \mathscr{B}_n\ (A <_\sigma B \Rightarrow \phi_0(A, B))$.

(vii) $\forall B \in \mathscr{B}_n\ \forall y \in Y_n\ (\{y\} <_\sigma B \wedge y \sim b \in B \Rightarrow f(\{y, b\}) = 0)$.

(viii) $\forall A, B \in \mathscr{B}_n (A \neq B \wedge a \in A \Rightarrow \Gamma(a) \cap B \in \mathscr{L} \wedge B \setminus \Gamma(a) \in \mathscr{L})$.

Conditions (iv) and (v) give the desired conclusion that (o) and (oo) hold.

In order to proceed from step n to step $n+1$ we apply ψ_0 to each $B \in \mathscr{B}_n$ in place of V to obtain two large subsets B_0, B_1 of B such that $B_0 <_\sigma B_1$ and $\phi_0(B_0, B_1)$. Let $\mathscr{B}_n^0 = \mathscr{B}_{n,0}^0 \cup \mathscr{B}_{n,1}^0$, where

$$\mathscr{B}_{n,0}^0 = \{B_0 : B \in \mathscr{B}_n\}, \quad \mathscr{B}_{n,1}^0 = \{B_1 : B \in \mathscr{B}_n\}.$$

Note that $\mathscr{B}_n^0$ is finite and linearly ordered under $<_\sigma$ and that whenever $A, B \in \mathscr{B}_n^0$ with $A <_\sigma B$ then $\phi_0(A, B)$. Apply Lemma 6 to the set $\mathscr{B}_n^0$ to obtain for each $B \in \mathscr{B}_n^0$ a large set $B' \subseteq B$ with the property that for all $A, B \in \mathscr{B}_n^0$ and all $x \in B'$ the sets $\Gamma(x) \cap A'$ and $A' \setminus \Gamma(x)$ are large. Put $\mathscr{B}_{n,i}^1 = \{B' : B \in \mathscr{B}_{n,i}^0\}$ $(i \in 2)$ and

$\mathscr{B}_n^1 = \mathscr{B}_{n,0}^1 \cup \mathscr{B}_{n,1}^1$. Note that the set $\mathscr{B}_n^1$ is still linearly ordered under $<_\sigma$ and whenever $A, B \in \mathscr{B}_n^1$ and $A <_\sigma B$, then $\phi_0(A,B)$.

Let $y \in \omega$ be a vertex such that there is a graph isomorphism $\alpha : \{\tau_0, \tau_1, \ldots, \tau_{n-1}, \tau_n\} \to \{\delta_0, \delta_1, \ldots, \delta_{n-1}, y\}$ given by $\forall i \in n\ (\alpha(\tau_i) = \delta_i)$ and $\alpha(\tau_n) = y$. Such a vertex $y \in \omega$ exists because of the mapping extension property of the Rado graph. Note that, for every $x \in \mathrm{Orb}(Y_n; y)$, the map $\alpha' : \{\tau_0, \tau_1, \ldots, \tau_{n-1}, \tau_n\} \to \{\delta_0, \delta_1, \ldots, \delta_{n-1}, x\}$ given by $\forall i \in n\ (\alpha'(\tau_i) = \delta_i)$ and $\alpha'(\tau_n) = x$ is also a graph isomorphism.

Let $C \in \mathscr{B}_n$ be the unique set such that $C \subseteq \mathrm{Orb}(Y_n; y)$. For each $i \in 2$ there is a unique set $C_i' \in \mathscr{B}_{n,i}^1$ such that $C_i' \subseteq C$ and $C_0' <_\sigma C_1'$. Since C_1' is a large subset of U, it follows by ψ_0 that there are two large sets $X \subseteq C_1'$ and $C_1'' \subseteq C_1'$ such that $X <_\sigma C_1''$ and $\phi_0(X, C_1'')$, and hence there is a large subset $X' \subseteq X$ such that for all $x \in X'$ the set $\Gamma_0(x) \cap C_1'' \in \mathscr{L}$.

If $Z \in (\mathscr{B}_{n,1}^1 \setminus \{C\}) \cup \{C_1''\}$ with $X' <_\sigma Z$ and if T is a large subset of X' then $\phi_0(T, Z)$. Hence there is a large subset $S \subseteq T$ such that for all $s \in S$ the set $\Gamma_0(s) \cap Z \in \mathscr{L}$. The set $\mathscr{B}_{n,1}^1$ is finite and hence there is a large set $X'' \subseteq X'$ such that for all $Z \in (\mathscr{B}_{n,1}^1 \setminus \{C\}) \cup \{C_1''\}$ with $X' <_\sigma Z$ and all $x \in X''$ we have $\Gamma_0(x) \cap Z \in \mathscr{L}$. By Corollary 3.5 there are only finitely many $x \in \omega$ such that the sequence $(\delta_0, \delta_1, \ldots, \delta_{n-1}, x)$ is not a subsequence of the sequence σ or such that $C_0' \backslash \Gamma(x) \notin \mathscr{L}$. Therefore, we can choose $\delta_n \in X''$ such that $\delta^{n+1} = (\delta_0, \delta_1, \ldots, \delta_{n-1}, \delta_n)$ is a subsequence of σ and such that $C_0' \backslash \Gamma(\delta_n) \in \mathscr{L}$. Notice that, since $C_0' <_\sigma X <_\sigma C_1''$, we have $C_0' <_\sigma \{\delta_n\} <_\sigma C_1''$. Put $Y_{n+1} = Y_n \cup \{\delta_n\}$.

Put $\mathscr{B}_{n,0}^2 = \mathscr{B}_{n,0}^1$, $\mathscr{B}_{n,1}^2 = (\mathscr{B}_{n,1}^1 \setminus \{C_1'\}) \cup \{C_1''\}$, and $\mathscr{B}_n^2 = \mathscr{B}_0^2 \cup \mathscr{B}_n^2$. Note that by the inductive hypothesis (viii), the construction of $\mathscr{B}_n^2$ and the choice of δ_n, we have that

$$\forall B \in \mathscr{B}_{n,0}^2\ (B \setminus \Gamma(\delta_n) \in \mathscr{L}),$$

$$\forall B \in \mathscr{B}_{n,1}^2\ (\Gamma(\delta_n) \cap B \in \mathscr{L}),$$

$$\forall B \in \mathscr{B}_{n,1}^2\ (\{\delta_n\} <_\sigma B \Rightarrow \Gamma_0(\delta_n) \cap B \in \mathscr{L}). \tag{3}$$

We are now ready to define the set $\mathscr{B}_{n+1}$. Put

$$F = \{\sigma_i : i < \sigma^{-1}(\delta_n)\},$$

$$\mathscr{D}_{n+1,0} = \{B \setminus (\Gamma(\delta_n) \cup F) : B \in \mathscr{B}_{n,0}^2\},$$

$$\mathscr{D}_{n+1,1}(<_\sigma) = \{B \cap \Gamma_0(\delta_n) \setminus F : B \in \mathscr{B}_{n,1}^2 \wedge B <_\sigma \{\delta_n\}\},$$

$$\mathscr{D}_{n+1,1}(>_\sigma) = \{B \cap \Gamma_0(\delta_n) \setminus F : B \in \mathscr{B}_{n,1}^2 \wedge B >_\sigma \{\delta_n\}\},$$

$$\mathscr{B}_{n+1} = \mathscr{D}_{n+1,0} \cup \mathscr{D}_{n+1,1}(<_\sigma) \cup \mathscr{D}_{n+1,1}(>_\sigma). \tag{4}$$

Clearly all the elements of $\mathscr{B}_{n+1}$ are subsets of U and they are large subsets of U by (3). The set $\mathscr{B}_{n+1}$ is linearly ordered under $<_\sigma$ and for all $A, B \in \mathscr{B}_{n+1}$, if $A <_\sigma B$, then $\phi_0(A, B)$. Let $(Y_{n+1}; x)$ be a type of the Rado graph $\mathscr{R} = (\omega; \sim)$. There is exactly one set $B \in \mathscr{B}_n$ such that

$$B \subseteq \mathrm{Orb}(Y_n; x).$$

There are exactly two sets $B'_0, B'_1 \in \mathscr{B}_{n+1}$ such that $B'_0, B'_1 \subseteq B$, and $B'_0 <_\sigma B'_1$. Note that δ_n is adjacent to no vertex in B'_0 and is adjacent to every vertex in B'_1. If the vertex x is not adjacent to the vertex δ_n then B'_0 is the unique set in $\mathscr{B}_{n+1}$ such that $B'_0 \subseteq \mathrm{Orb}(Y_{n+1}; x)$, and if the vertex x is adjacent to the vertex δ_n then B'_1 is the unique set in $\mathscr{B}_{n+1}$ such that $B'_1 \subseteq \mathrm{Orb}(Y_{n+1}; x)$. It is clear from the construction that conditions (i), (ii), (vi) and (viii) hold with n replaced by $n+1$. We have to check that the remaining conditions also hold for $n+1$. By the construction there are $C \in \mathscr{B}_n$, $C'_0 \in \mathscr{B}^2_{n,0}$, $C''_1 \in \mathscr{B}^2_{n,1}$ such that $\delta_n \in C$ and $C'_0 <_\sigma \{\delta_n\} <_\sigma C''_1$.

(iii) Since δ_n was chosen as an element of one of the members of $\mathscr{B}_n$, it is immediate from the induction hypothesis and the construction of the set $\mathscr{B}_{n+1}$, that $\mathscr{B}_{n+1} \cup Y_{n+1}$ is totally ordered under $\leq_\sigma$. To prove that (iii) holds for $n+1$, we will prove that, whenever $X, Y \in \mathscr{B}_{n+1} \cup Y_{n+1}$ are such that $X <_\sigma Y$ holds, then $X <_{n+1} Y$. We distinguish several subsequences.

Case 1. $y \in Y_{n+1}$ and $B \in \mathscr{B}_{n+1}$. Because $a <_n b \Rightarrow a <_{n+1} b$ and the elements of $\mathscr{B}_{n+1}$ are subsets of elements of $\mathscr{B}_n$, it follows that property (iii) holds for all $y \in Y_n$ and all $B \in \mathscr{B}_{n+1}$. Hence we may assume that $y = \delta_n$. Suppose $B \not\subseteq C$. There is $B^* \in \mathscr{B}_n$ be such that $B \subseteq B^*$. By the induction hypothesis, if $C <_\sigma B^*$ then $C <_n B^*$ and because $\delta_n \in C$, we get $\{\delta_n\} <_n B$. Similarly, if $B^* <_\sigma C$ then $B <_n \{\delta_n\}$. Suppose $B \subseteq C$. Then either $B = C'_0$ or $B = C''_1$. In either subsequence, since $\delta_n \in C$, $(\mathrm{lex}(b, \delta^n) = \mathrm{lex}(\delta_n, \delta^n))$ for all $b \in B$. If $b \in C'_0$ then $\delta_n \sim b$ and $\delta_n \sim \delta_n$ and hence $b <_{n+1} \delta_n$. Thus $B <_{n+1} \{\delta_n\}$ if $B \subseteq C'_0$. If $b \in C''_1$ then $\delta_n \sim b$, and so $\mathrm{lex}(b, \delta^{n+1})$ is a proper extension of $\mathrm{lex}(\delta_n, \delta^{n+1})$ which ends in a 1. This implies that $\delta_n <_{n+1} b$ and hence $\{\delta_n\} <_{n+1} B$ if $B \subseteq C''_1$.

Case 2. A, $B \in \mathscr{B}_{n+1}$. Suppose $A <_\sigma B$ holds. There are sets $A^*, B^* \in \mathscr{B}_n$ with $A \subseteq A^*$ and $B \subseteq B^*$. If $A^* \neq B^*$, then $A^* <_\sigma B^*$ and hence $A^* <_n B^*$. Since $a <_n b$ implies $a <_{n+1} b$, it follows that $A <_{n+1} B$. Suppose $A^* = B^* = D$. In this case $\mathrm{lex}(a, \delta_n) = \mathrm{lex}(b, \delta^n)$ holds for all $a \in A$ and $b \in B$. Since $A <_\sigma B$, it follows that $A \in \mathscr{D}_{n+1,0}$ and $B \in \mathscr{D}_{n+1,1}(<_\sigma) \cup \mathscr{D}_{n+1,1}(>_\sigma)$. According to definition (3) the vertex δ_n is not adjacent to any vertex in A but adjacent to all vertices in B. This implies that $A <_{n+1} B$.

Case 3. a, $b \in Y_{n+1}$. Assume that $a <_\sigma b$. If neither a, b is equal to δ_n, then by the induction hypothesis $a <_n b$, which implies $a <_{n+1} b$. Therefore, we may assume that one of a and b is δ_n. Suppose $a = \delta_n$ and $b \in Y_n$. Then $a <_\sigma b$ implies $C <_\sigma \{b\}$, and hence $C <_n \{b\}$. Therefore, $a <_n b$ and so $a <_{n+1} b$. Similarly if $b = \delta_n$.

(iv) It follows from the induction hypothesis and the choice of δ_n that the function $\alpha_n:\{\tau_0,\tau_1,\dots,\tau_n\}\to\{\delta_0,\delta_1,\dots,\delta_n\}$ given by $\alpha(\tau_i)=\delta_i$ is a graph isomorphism. Also, if $i,j\in n$ and $\tau_i<_\tau\tau_j$ then $\delta_i<_n\delta_j$ and hence $\delta_i<_{n+1}\delta_j$. It follows that $\delta_i<_\sigma\delta_j$ by (iii). Suppose $i\in n$ and $\tau_i<_\tau\tau_n$ then $\delta_i<_n\delta_n$ because α is a graph isomorphism. Therefore, $\delta_i<_{n+1}\delta_n$ and $\delta_i<_\sigma\delta_n$ by (iii). Similarly if $\tau_i>_\tau\tau_n$.

(v) We only have to check property (v) for the case when $i\in n$, $\delta_i\sim\delta_n$ and $\delta_i<_\sigma\delta_n$. Since $\delta_i<_\sigma\delta_n$, it follows that $\{\delta_i\}<_\sigma C$. It then follows then from property (vii) for n that $f(\{\delta_i,\delta_n\})=0$.

(vii) Suppose that $y\in Y_n$, $b\in B\in\mathscr{B}_{n+1}$, and that $\{y\}<_\sigma B$ and $y\sim b$. There is exactly one $B^*\in\mathscr{B}_n$ with $B\subseteq B^*$. Since $b\in B^*$ and $\{y\}<_\sigma B^*$ and $y\sim b$, it follows from the induction hypothesis that $f(\{y,b\})=0$. Now assume that $b\in B\in\mathscr{B}_{n+1}$ and that $\{\delta_n\}<_\sigma B$ and $\delta_n\sim b$. Then $B\in\mathscr{D}_{n+1,1}(>_\sigma)$ and hence from definition (3) it follows that $f(\{\delta_n,b\})=0$. ∎

5. The results

Lemma 5.1. *Let $\mathscr{R}_\sigma$ and $\mathscr{R}_\tau$ be lexicographically ordered Rado graphs. Then there exists a subsequence δ of σ such that the function α given by $\alpha(\tau_i)=\delta_i$ is an order reversing graph isomorphism, that is $\tau_i\sim\tau_j\Leftrightarrow\delta_i\sim\delta_j$ and $\tau_i<_\tau\tau_j\Leftrightarrow\delta_i>_\sigma\delta_j$.*

Proof. We will inductively construct three subsequences δ'',δ',δ of the sequence σ as follows.

Put $\delta''_0=\sigma_0$, $\delta'_0=\sigma_1$ and, using the mapping extension property, choose for δ_0 some vertex in ω such that $\delta''_0\not\sim\delta_0\wedge\delta'_0\sim\delta_0\wedge\sigma^{-1}(\delta_0)>1$. Now suppose that $n>0$ and that δ''_i, δ'_i and δ_i have been chosen for $i<n$. Then choose δ''_n, δ'_n to be the next two terms in σ after δ_{n-1}, i.e. so that $\sigma^{-1}(\delta''_n)=\sigma^{-1}(\delta_{n-1})+1$, $\sigma^{-1}(\delta'_n)=\sigma^{-1}(\delta_{n-1})+2$. Finally choose δ_n so that:

$$\begin{gathered}\sigma^{-1}(\delta_n)>\sigma^{-1}(\delta'_n),\\ \forall i\le n\ (\delta_i\sim\delta_n\Leftrightarrow\tau_i\sim\tau_n),\\ \forall i\le n\ (\delta''_i\sim\delta_n\Leftrightarrow\delta_i\not\sim\delta_n),\\ \forall i\le n\ (\delta'_i\sim\delta_n\Leftrightarrow i=n),\\ \forall j\le\sigma^{-1}(\delta'_n)\ (\sigma_j\notin\bigcup_{i\le n}\{\delta''_i,\delta'_i,\delta_i\}\Rightarrow\sigma_j\not\sim\delta_n).\end{gathered}$$

By the mapping extension property of the Rado graph it is easy to see that there is a vertex δ_n satisfying the above conditions.

It is clear from the construction that the map α with $\alpha(\tau_i) = \delta_i$ is a graph isomorphism, and hence $\tau_i \leq_\tau \tau_j \Leftrightarrow \delta_i \leq_\delta \delta_j$. It suffices therefore to prove that $\forall i$, $j \in \omega$ $(\delta_i <_\delta \delta_j \rightarrow \delta_i >_\sigma \delta_j)$. Assume $\delta_i <_\delta \delta_j$.

Case 1: $m = \min\{l : \delta_i \not\sim \delta_l \wedge \delta_j \sim \delta_l\} < \min\{i,j\}$. Then $\forall k < \sigma^{-1}(\delta''_m)$ $(\delta_i \sim \sigma_k \Leftrightarrow \delta_j \sim \sigma_k)$ but $\delta_i \sim \delta''_m \wedge \delta_j \not\sim \delta''_m$ because $\delta_i \not\sim \delta_m \wedge \delta_j \sim \delta_m$. Hence $\delta_i >_\sigma \delta_j$.

Case 2: $\min\{l : \delta_i \not\sim \delta_l \wedge \delta_j \sim \delta_l\} = i < j$. This is impossible since $\delta_i \sim \delta_i$.

Case 3: $i < j \wedge \forall l \leq i$ $(\delta_i \sim \delta_l \Leftrightarrow \delta_j \sim \delta_l)$. Then $\forall k \leq \sigma^{-1}(\delta''_i)$ $(\delta_i \sim \sigma_k \Leftrightarrow \delta_j \sim \sigma_k)$. But $\delta_i \sim \delta'_i \wedge \delta_j \not\sim \delta'_i$, and hence $\delta_i >_\sigma \delta_j$.

Case 4: $j = \min\{l : \delta_i \not\sim \delta_l \wedge \delta_j \sim \delta_l\} < i$. Then $\forall k < \sigma^{-1}(\delta''_j)$ $(\delta_i \sim \sigma_k \Leftrightarrow \delta_j \sim \sigma_k)$ but $\delta_i \sim \delta''_j \wedge \delta_j \not\sim \delta''_j$ because $\delta_i \not\sim \delta_j \wedge \delta_j \sim \delta_j$. Hence $\delta_i >_\sigma \delta_j$. ∎

Theorem 5.2. *Let $\mathcal{R}_\sigma$ and $\mathcal{R}_\tau$ be lexicographically ordered Rado graphs, and let $r \in \omega$. Then*

$$\mathcal{R}_\sigma \rightarrow (\mathcal{R}_\tau)^<_r \tag{5}$$

and

$$\mathcal{R}_\sigma \rightarrow (\mathcal{R}_\tau)^>_r \tag{6}$$

Proof. It is clearly enough to prove the theorem for the case $r = 2$. In this case (5) is then immediate from Lemma 7. We shall derive (6) for $r = 2$ from this.

Let $f : E_\sigma(>) \rightarrow 2$ be any 2-colouring. We will show that there are $\ell \in 2$ and an increasing map $\vartheta : \omega \rightarrow \omega$ such that the map $\tau_i \mapsto \sigma_{\vartheta(i)}$ is an embedding of $\mathcal{R}_\tau$ into $\mathcal{R}_\sigma$ such that

$$\forall i,\ j \in \omega \quad (i < j \wedge \tau_i \sim \tau_j \wedge \tau_i >_\tau \tau_j \Rightarrow f(\{\sigma_{\vartheta(i)}, \sigma_{\vartheta(j)}\}) = \ell).$$

By Lemma 5.1 there is an increasing map $\alpha : \omega \rightarrow \omega$ such that the map $\sigma_i \mapsto \sigma_{\alpha(i)}$ is an order reversing graph isomorphism of R_σ into itself, i.e.

$$\forall i,\ j \in \omega \quad [(\sigma_i \sim \sigma_j \Leftrightarrow \sigma_{\alpha(i)} \sim \sigma_{\alpha(j)}) \ \wedge\ (\sigma_i <_\sigma \sigma_j \Leftrightarrow \sigma_{\alpha(i)} >_\sigma \sigma_{\alpha(j)})].$$

Let f^* be the 2-colouring of $E_\sigma(<)$ defined by

$$f^*(\sigma_i, \sigma_j) = f(\{\sigma_{\alpha(i)}, \sigma_{\alpha(j)}\})$$

for $i < j$ and $\sigma_i <_\sigma \sigma_j$. By (5), there are $\ell \in 2$ and an increasing map $\beta : \omega \rightarrow \omega$ such that

$$\forall i, j \in \omega\ [(\tau_i \sim \tau_j \Leftrightarrow \sigma_{\beta(i)} \sim \sigma_{\beta(j)}) \ \wedge\ (\tau_i <_\tau \tau_j \Leftrightarrow \sigma_{\beta(i)} <_\sigma \sigma_{\beta(j)})],$$

and

$$\forall i, j \in \omega\ (i < j \wedge \tau_i \sim \tau_j \wedge \tau_i <_\tau \tau_j \Rightarrow f^*(\sigma_{\beta(i)}, \sigma_{\beta(j)}) = \ell).$$

By Lemma 5.1 again, there is an increasing map $\gamma : \omega \to \omega$ such that the map $\tau_i \mapsto \tau_{\gamma(i)}$ is an order reversing graph isomorphism of R_τ into itself, i.e.

$$\forall i,j \in \omega\ [(\tau_i \sim \tau_j \Leftrightarrow \tau_{\gamma(i)} \sim \tau_{\gamma(j)}) \ \wedge\ (\tau_i <_\tau \tau_j \Leftrightarrow \tau_{\gamma(i)} >_\tau \tau_{\gamma(j)})].$$

Put $\vartheta = \alpha \circ \beta \circ \gamma$. Clearly ϑ is an order preserving graph isomorphism of $\mathcal{R}_\tau$ into $\mathcal{R}_\sigma$ since

$$\forall i,j \in \omega\ (\tau_i \sim \tau_j \Leftrightarrow \tau_{\gamma(i)} \sim \tau_{\gamma(j)} \Leftrightarrow \sigma_{\beta(\gamma(i))} \sim \sigma_{\beta(\gamma(j))} \Leftrightarrow \sigma_{\vartheta(i)} \sim \sigma_{\vartheta(j)}),$$

and whenever $i<j$

$$\tau_i <_\tau \tau_j \Leftrightarrow \tau_{\gamma(i)} >_\tau \tau_{\gamma(j)} \Leftrightarrow \sigma_{\beta(\gamma(i))} >_\sigma \sigma_{\beta(\gamma(j))} \Leftrightarrow \sigma_{\vartheta(i)} <_\sigma \sigma_{\vartheta(j)}.$$

Now suppose that $i<j$, $\tau_i \sim \tau_j$ and $\tau_i >_\tau \tau_j$. Then $\tau_{\gamma(i)} \sim \tau_{\gamma(j)}$ and $\tau_{\gamma(i)} <_\tau \tau_{\gamma(j)}$ and so $f^*(\sigma_{\beta(\gamma(i))}, \sigma_{\beta(\gamma(j))}) = \ell$, and hence $f(\{\sigma_{\vartheta(i)}, \sigma_{\vartheta(j)}\}) = \ell$. ∎

Theorem 5.3. *Let $f : [\omega]^2 \to r \in \omega$ be an r-colouring of the pairs of vertices of the Rado graph $\mathcal{R}$ on ω. Then there is an increasing map $\alpha : \omega \to \omega$ such that $\mathcal{R}|\alpha(\omega)$ is an induced copy of $\mathcal{R}$ and $|f([\alpha(\omega)]^2)| \leq 4$.*

Proof. The set of vertices of $\mathcal{R}$ is ω and hence $\mathcal{R}$ can be viewed as a lexicographically ordered Rado graph with the natural ordering of the vertices. It follows from Theorem 5.2 that there is an order preserving isomorphic copy of $\mathcal{R}$ in itself in which the edges have at most two colours. Also, since the graph obtained from the Rado graph by interchanging edges and non-edges is also a Rado graph, it follows that there is an order preserving isomorphic copy of $\mathcal{R}$ in itself in which the edges have at most two colours and the non-edges also have at most two colours. ∎

It follows from Theorem 5.2 that for every coloured Rado graph $\mathcal{R} \in \mathbf{R}_r$ there is a basic coloured Rado graph $\mathcal{B}$ which embeds into $\mathcal{R}$ and by Lemma 4.1 any two basic coloured Rado graphs which use the same set of colours are equimorphic. According to [4] no coloured Rado graph whose edges are coloured with only one colour can be embedded into a basic coloured Rado graph which uses two different colours. Of course two basic coloured Rado graphs which use a different set of colours can not be embedded into each other. Hence we can conclude with the following theorem.

Choose for every $r \in \omega$ and every pair $(c_0, c_1) \in r^2$ with $c_0 \leq c_1$ a (c_0, c_1)-coloured basic Rado graph $\mathcal{B}_{(c_0,c_1)}$ and put $\mathbf{B}_r = \{\mathcal{B}_{(c_0,c_1)} : (c_0,c_1) \in r^2 \wedge c_0 \leq c_1\}$.

Theorem 5.4. *The set $\mathbf{B}_r$ is a Ramsey basis of the set $\mathbf{R}_r$.* ∎

Acknowledgement. We are both very grateful to E. C. Milner for rewriting our final version of this paper.

References

[1] R. FRAÏSSÉ: Theory of Relations, *Studies in Logic and the Foundations of Mathematics,* Elsevier Science Publishing Co., Inc., U.S.A., **118** (1986), 313.

[2] S. TODORCEVIC: Some partitions of three-dimensional combinatorial cubes, *Journal of Combinatorial Theory A,* **68** (1994), 410–437.

[3] D. DEVLIN: *Some partition theorems and ultrafilters on ω,* Ph. D. Thesis, Dartmouth College, 1979.

[4] P. ERDŐS, A. HAJNAL, L. PÓSA: Strong embeddings of graphs into coloured graphs, *Colloquia Mathematica Societatis János Bolyai,* **10, Vol. I**, Infinite and finite sets, 585–595, Edited by A. Hajnal, R. Rado, Vera T. Sós.

Maurice Pouzet

Laboratoire d'Algèbre Ordinale,
Institut de Mathématiques
et Informatique, ISM,
Université Claude Bernard Lyon 1,
69622 Villeurbanne Cedex, France
pouzet@sunlmdi.univ-lyon1.fr

Norbert Sauer

Department of Mathematics and Statistics
The University of Calgary
Calgary, T2N 1N4, Alberta, Canada
nsauer@acs.ucalgary.ca